Reliability analysis with consideration of asymmetrically dependent variables: discussion and application to geotechnical examples

Yi Zhang[a,b]*,*António Topa Gomes*[c,d], *Michael Beer*[d], *Ingo Neumann*[b], *Udo Nackenhorst*[e], *Chul-Woo Kim*[f]

[a]Department of Civil Engineering, Tsinghua University, China

[b] Geodetic Institute, Leibniz Universität Hannover, Germany

[c] CONSTRUCT, Faculty of Engineering (FEUP), University of Porto, Portugal

[d] Institute for Risk and Reliability, Leibniz Universität Hannover, Germany

[e] Institute of Mechanics and Computational Mechanics,Leibniz Universität Hannover, Germany

[f] Department of Civil and Earth Resources Engineering, Graduate School of Engineering, Kyoto University, Japan

Abstract

The consideration of multivariate models in the reliability analysis is quite essential from practical perspective. In principle, complete information regarding the joint probability distribution function should be known in prior to the analysis. However, in real practice, only the marginal distribution and covariance matrix are known in most cases. Such incomplete probabilistic information could lead to dubious results if dependences are not fully catered. Asymmetric dependence is one of these factors influencing the quality of reliability analysis. In this paper, the influences of asymmetric dependences to the reliability problem are investigated. The copula theory as well as the concept of asymmetric dependences is briefly introduced. The techniques of constructing asymmetric copulas are, thereafter, provided in details. Geotechnical problem is selected in this study as examples in the reliability analysis. Based on the given information, a group of symmetric and asymmetric copulas are selected to model the dependences between cohesion and friction angle, the parameters more commonly used to characterize soil strength. The reliability analysis of a

* Corresponding author. Tel.: +81-075-383-3421
E-mail address: zhang_yi87@163.com; (Y. Zhang).

continuous spread footing and an infinite slope are then presented to demonstrate the influence of asymmetric dependences on reliability. The results showed that the failure probabilities of the investigated geotechnical problems are very sensitive to the adopted dependence structure, either symmetrically or asymmetrically. The commonly applied one parameter symmetric copulas, such as Archimedean copulas, may underestimate the failure probabilities. Furthermore, the asymmetric copulas are more powerful in characterizing the tail dependences structures of variables especially for asymmetric dependent variables.

Keywords: reliability analysis, joint distribution, multivariate analysis, asymmetric copula, geotechnical engineering

1. Introduction

Reliability analysis is frequently associated with multivariate data analysis. To achieve an accurate estimation of the reliability problem, an adequate joint probabilistic distribution function of the variables is required. However, in most of the engineering practice, the full information, including the marginal distribution and dependences between the engineering parameters, cannot be determined. Usually, only the marginal distributions and covariance matrix are known. In this context, the modeling of the dependences among parameters plays an important role in the reliability analysis. Deficiencies in modeling their joint relationship may lead to large errors estimating the failure probability of reliability problems, hence leading to expensive losses (Phoon and Kulhawy, 1999, Beer et al., 2013).

The problem associated with dependences is particularly critical in geotechnical engineering as geotechnical parameters are frequently observed to be dependant in real practice. For instance, the shear strength parameters, cohesion and tangent of friction angle, are found to be negatively correlated in most cases (Pinheiro Branco et al., 2014). The soil test results like standard penetration test (SPT) and piezocone test (CPTU) are believed to be physically related (Robertson, 2009). The key problem in characterizing this relationship is how to define the word “dependence”. The typical word “dependence” in this context can be related to various kinds of meanings in real practice. Usually, the concept of correlation is utilized as the most common idea in characterizing the dependences among soil parameters in practice. The simplicity of this concept has made its use widely spread in the engineering applications. For example, the Nataf distribution is widely employed in geotechnical engineering field for constructing the joint distributions of soil parameters based on their correlations (Li et al., 2015). However, this concept was also criticized for its limitation in measuring only the linear dependence and found to be inaccurate in describing soil parameters having complex dependences (Wang and Li, 2018). It was also noted the correlation based joint distribution produces only one of the various possible solutions of failure probabilities for the geotechnical problem and such a probability may be biased towards the unconservative side (Phoon and Ching, 2015). Nevertheless, many recent works were published devoting to the presentation of multivariate information (Ching & Phoon, 2014; Zhang et al., 2018a).

Compared to the traditional joint models, copula was found to be very popular and attracted significant attention of engineering researchers (Wu 2013, Tang et al., 2015). A prominent feature of copula model is its flexibility in modeling the dependence structure, which can be separated from the modeling of individual behavior. For geotechnical problems, this characteristic is highly desirable as most soil data exhibit nonlinear dependencies. It was found the use of copula could improve the quality of reliability analysis of an engineering problem (Li et al., 2012). However, there still exist various types of complex dependences which are not well characterized by a normal copula model. Among these, the asymmetric dependence is one of the most complicated dependences that need to be paid attention to. Asymmetric dependences are referring to the dependence structures having unequal upper-lower and lower-upper tail dependences. In reliability problems, the asymmetric dependences among variables can be frequently observed in various cases especially for geotechnical engineering. For example, the soil parameter undrained shear strength, preconsolidation stress and vertical effective stress are usually believed to be asymmetrically dependant with each other. The reason is that they are inherently dependent on the liquid limit and over consolidation ratio which are not a direct influencing factor that makes their dependences quite asymmetric. There are several other paired ground parameters which also possesses certain degree of asymmetric dependences, as is the case of void-ratio and unit weight, unit weight and dry unit weight, void ratio and dilation angle, etc. One common reason of such asymmetric dependences among these soil parameters is due to the physical limitations. That is, the occurrence for some data combinations is physically not possible. All these combinations impact the reliability analysis although less importance than the strength parameters. Nevertheless, the influences of asymmetric dependences to the reliability of geotechnical problems have never been studied in detail. The impact of uncertainties in the asymmetric dependences for soil data to the overall geotechnical problem assessment has not yet been investigated. Therefore, this work aims to fill in this gap by presenting a real case study for asymmetric dependences, highlighting the influences of adopting different asymmetric copulas in the reliability analysis. Since geotechnical engineering has more practices related to the asymmetric dependence problems, in this study, we choose to utilize the geotechnical problems as example for the investigation. However, the results from this study will be interpreted based on general reliability engineering perceptions.

This paper contains four sections. A general review of the copula theory and the concept of asymmetric dependences are discussed in Section 2. Section 3 then introduces the procedures of constructing asymmetric copula and its flexibility in characterizing the dependences. Two geotechnical examples are then analyzed through the use of asymmetric copulas in modeling the soil parameters. Section 4 provides the detailed discussion on the analysis and results. A comparison is made in the investigation between the use of symmetric and asymmetric copulas. The conclusions drawn from this study are summarized in Section 5.

2. Copula theory and the fact of asymmetric dependence

As mentioned previously, copula models provide a very flexible way of modeling the multivariate dependences. Because of its high applicability, it has already been applied to a wide range of engineering applications including, for example, offshore engineering (see, Noh et al., 2009; Zhang et al., 2015; Wang et al., 2017), reliability engineering (Zhang and Lam, 2016; He et al. 2018), hydrology (Salvadori and De Michele, 2007) as well as economics (Fan and Patton, 2014; Zhang, 2018). The theoretical background has already been established by the former researchers, see Appendix A. For the reference of already developed copula models, one can refer to Nelsen (2006) and Joe (2014).

Nevertheless, the traditional copulas (e.g. Archimedean copulas) may have problems when they are used in engineering practices. Specifically, the traditional copulas can hardly capture the asymmetric dependences in the data sample. Unfortunately, these asymmetric dependences commonly exist in engineering practice, e.g. geotechnical designs. For example, the feasible domain of soil parameters is usually quite restricted because of the physical phenomenon. This is also a major reason causing asymmetric dependencies among most engineering variables. A typical example would be the soil cohesion strength and soil friction angle. It is impossible to have a large value of soil cohesion strength accompanied by a large value of friction angle because of the physical limit. Therefore, the observations of some variable combinations could not exist in real nature. This effect is illustrated through an example scatter plot in Fig. 1. As seen in the figure, the lower-right region (marked with a cross) contains no data. The scatterings of the data can only be available in the left-upper region (marked with a tick). More typical examples can be seen from the scatter plot of soil data retrieved from the database provided in the webpage of the Technical Committee on Risk Assessment & Management (TC304) in Fig. 2. As illustrated in the plots, the scatterness of the chosen soil parameters undrained shear strength s_u, preconsolidation stress σ'_p and vertical effective stress σ'_v are not symmetric. In fact, they are inherently dependent on the liquid limit and over consolidation ratio which makes their dependences quite asymmetric. From these scatter plots, it can be seen obviously that no data is distributed in the upper-lower domain (as marked by the red star symbol).

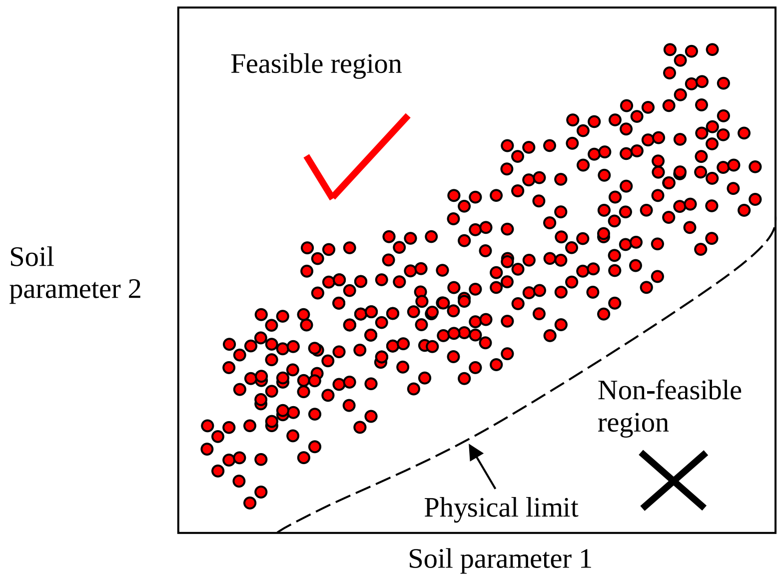
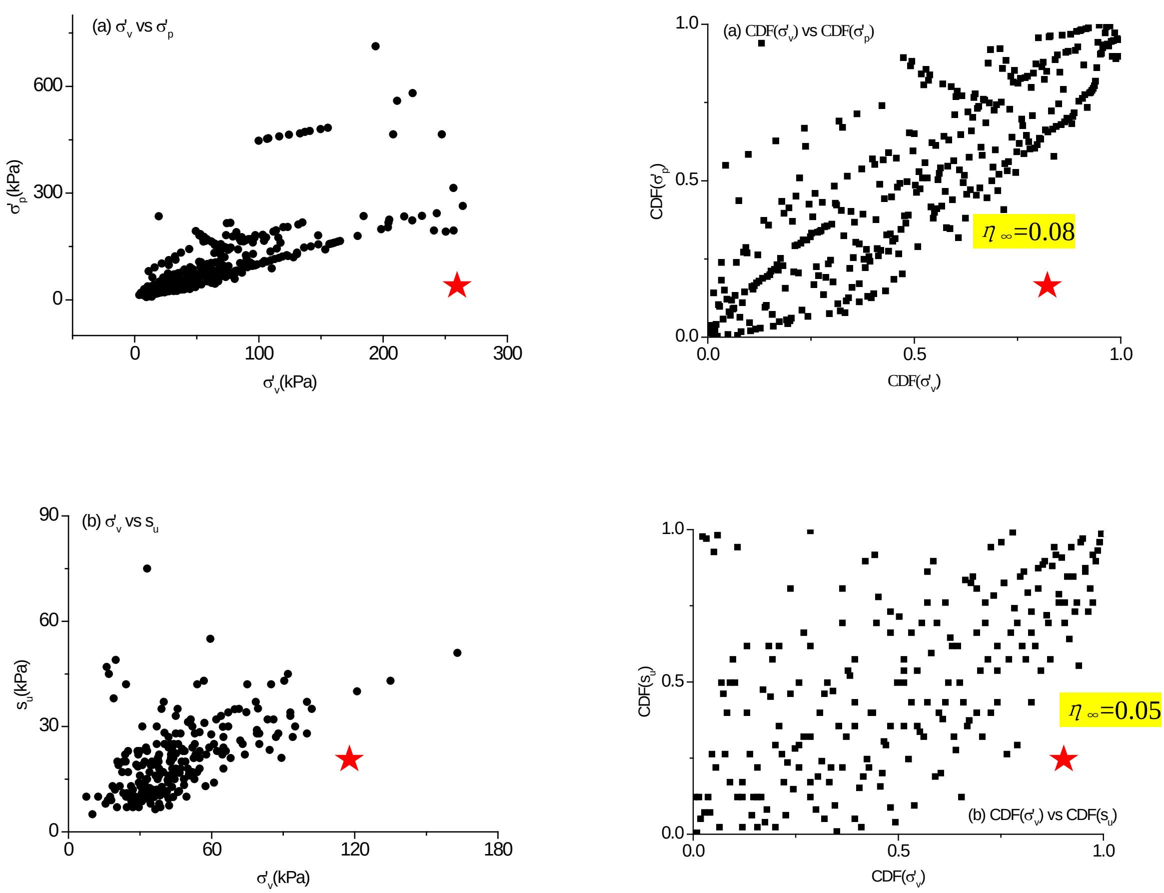


Figure **1** Asymmetric domain caused by physical phenomenon.



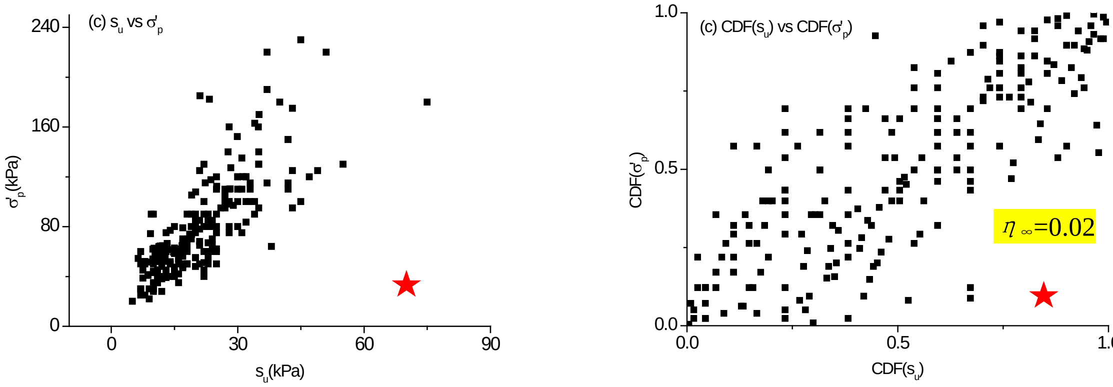


Figure 2 Examples of soil data having asymmetric domain (data retrieved from Ching and Phoon, (2012), Ching et al. (2014) and D'Ignazio et al. (2016))

This implicit physical phenomenon could exert limit of occurrence for some data combinations, which reduces the feasible domain of the variables. Concerning these physical features in the multivariate data modeling, especially copula approach, is not straightforward and still needs further development. More advanced techniques are therefore needed on the improvement of traditional copula model to further enhance this approach.

3. Asymmetric copulas

To capture the asymmetric dependences in a copula function, the technique of constructing asymmetric copulas is introduced herein. For the measure of asymmetric dependence, one can refer to Appendix A.

3.1 Formulations of asymmetric copulas

To cater for the asymmetric dependences in a copula, the technique of asymmetrizing is needed. In other words, the construction of asymmetric copulas is based on a combination of initial existing copulas and the procedures of asymmetrizing. Various ways of constructing asymmetric copulas have been studied in the prior works (Grimaldi and Serinaldi, 2006; Mesiar and Najjari, 2014; Mazo et al., 2015). However, not all these former developed asymmetric copulas are useful in practice. Many of them need very sophisticated extra functions to characterize the asymmetric dependencies, which are quite cumbersome for the numerical computations. For example, the Archimax copula developed by Charpentier et al. (2014) is an asymmetric copula that requires the Pickhands dependence function for its construction. Therefore, from the practical point of view, the most commonly applied asymmetrizing technique is discussed herein. Meanwhile, this work is devoted to the construction of asymmetric copula families based on the traditional symmetric copulas, e.g. Archimedean copulas. Therefore, the asymmetric copulas with a very complicated mathematical formulation would not be the primary concern in this study.

The most popular and simple way of constructing asymmetric copulas is by means of the Khoudraji transformations (Liebscher, 2008). Through such modification, the traditional Archimedean copulas can be asymmetrized. The general formula for constructing this kind of asymmetric copula is given as following

$$C_K\left(u_1,\dots,u_n\right)=\prod_{i=1}^{m} C_i\left(g_{i1}\left(u_1\right),\dots,g_{in}\left(u_n\right)\right)\ , \tag{1}$$

where C_K is the constructed asymmetric copula based on Khoudraji transformation, $C_1,\dots,C_m$ are the base copulas which are for n-dimensional variables, g_{ij}: $[0,1] \rightarrow [0,1]$ for i=1,…,m, j=1,…,n are the individual functions which should be strictly increasing or identically equal to 1. The individual functions here play an important role in asymmetrizing the copulas. The formulation of the individual functions g_{ij} need to follow very strict rules in order to guarantee the fundamental properties of copula. The following conditions must be satisfied:

1. $g_{ij}(1) = 1$ and $g_{ij}(0) = 0$,
2. g_{ij} is continuous on $[0,1]$,
3. If there are at least two functions $g_{i_1 j}, g_{i_2 j}$ with $1 \leq i_1, i_2 \leq m$ which are not identically equal to 1, then $g_{ij}(x) > x$ holds for $x \in (0,1)$, i=1,…,m.

From the above formulation, it is easy to see the properties of constructed asymmetric copula are largely dependent on the individual functions. This asymmetrizing technique is also known as an extension of Khoudraji's device (1995).On the other hand, it should also be realized various groups of parametric copulas can be selected for the base copulas $C_1, \dots, C_m$, e.g. Archimedean copulas. As for the individual functions g_{ij}, many candidate functions which are suitable for the copula construction have been proposed by Liebscher (2008) - see Table 1. One should know, by adopting type I individual function in Table 1 and setting *m*, *n*=2, Eq. (1) becomes exactly the Khoudraji copula. Moreover, it is also possible to choose the number and type of individual copulas. Such flexibility has made this asymmetric copula able to be extended to more complex multivariate models. A general procedure of modeling the multivariate data by using asymmetric copulas is illustrated in a flow chart in Fig. 3.

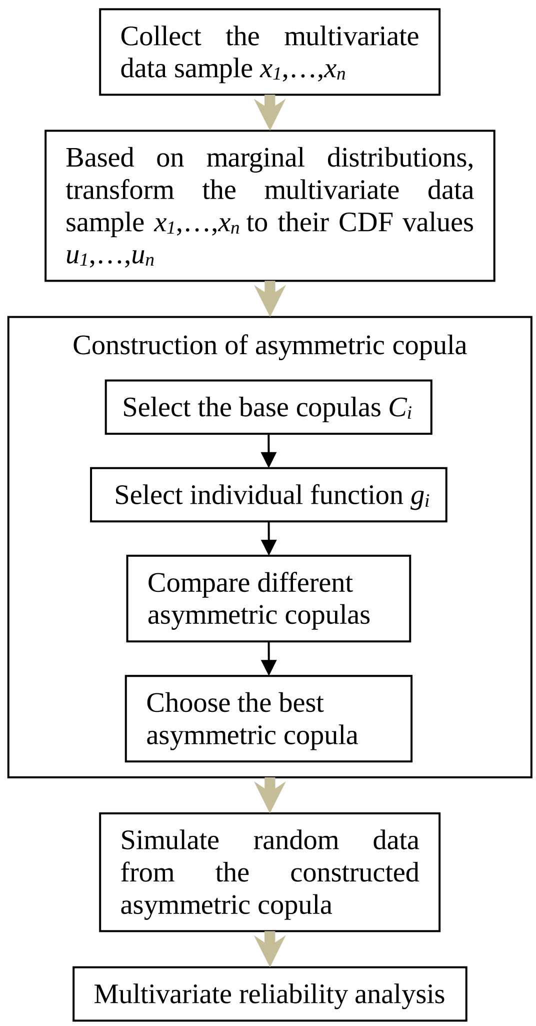


Figure 3 Flowchart of reliability analysis for asymmetrically dependant variables.

Table 1 Examples of individual functions

Individual function	Parameters	Value range

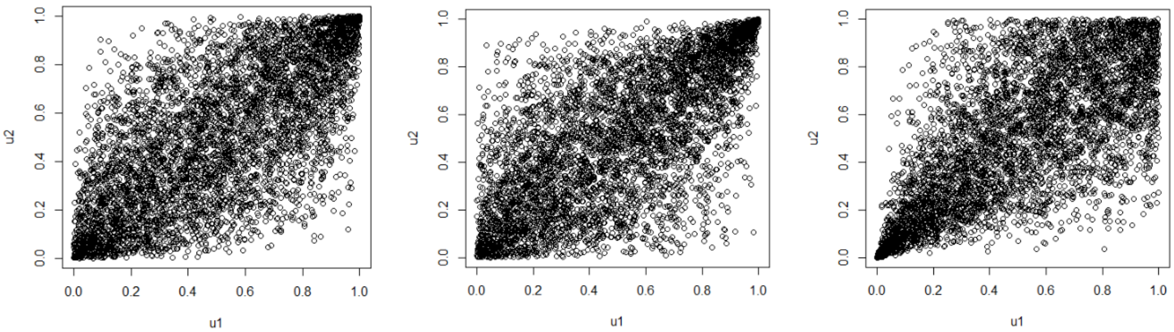
I.	$g_{ij}\left(u\right)=u^{\theta_{ij}}$	$\sum_{i=1}^{m}\theta_{ij}=1$	$\theta_{ij}\in\left[0,1\right]$
II.	$g_{ij}\left(u\right)=u^{\theta_{ij}}e^{(u-1)\alpha_{ij}}$	$\sum_{i=1}^{m}\theta_{ij}=1$, $\sum_{i=1}^{m}\alpha_{ij}=0$	$\theta_{ij}\in\left(0,1\right)$, $\alpha_{ij}\in\left(-\infty,1\right)$, $\theta_{ij}+\alpha_{ij}\geq 0$
III.	$*\ g_{1j}\left(u\right)=\exp\left(\theta_j-\sqrt{\left\|\ln u\right\|+\theta_j^2}\right)$, $g_{2j}\left(u\right)=u\exp(-\theta_j+\sqrt{\left\|\ln u\right\|+\theta_j^2})$	θ_j *for* $j\in\left\{1,\ldots,n\right\}$	$\theta_j\geq\frac{1}{2}$

*Note: type III individual functions can only be used for the asymmetric copula having two individual copulas (e.g. *m*=2).

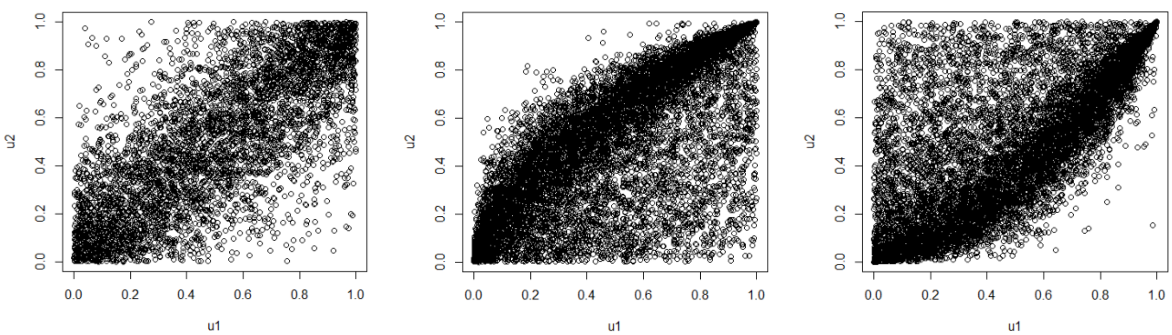
3.1 Comparison between asymmetric copulas and traditional copulas

To have a general sense of the asymmetric copulas, a comparison between the traditional copulas and asymmetric copulas is presented herein. The scatter plot for bivariate data having dependences following traditional copulas including Gaussian, Gumbel, Clayton and Frank is compared in Fig. 4. For demonstrating purpose, two asymmetric copulas, which are constructed by using two base copulas, Gumbel copula and Clayton, are also included in the comparison. The type I individual function in Table 1 is utilized in this asymmetric copula construction. The parameter values of the individual function are set at (θ_{11}=0.3, θ_{12}=0.6) and (θ_{11}=0.6, θ_{12}=0.3) for each of the asymmetric copulas. To make an acceptable comparison, the Spearman's ρ_s of all the bivariate data simulated from these copulas is set to be 0.7. From the scatter plot results that each copula characterizes a specific type of dependences. Compared to the traditional copulas, the asymmetric phenomenon in the dependence of the bivariate data can be obviously observed in the asymmetric copula examples. In the traditional copula examples, although the dependences can be diversified (as shown in the scatterings concentrations), the data can only be distributed symmetrically with the diagonal line. In other words, the lower-upper tail dependence $\lambda^{l,u}$ equals to the upper-lower tail dependence $\lambda^{u,l}$ in the symmetric copula, whereas $\lambda^{l,u}\neq\lambda^{u,l}$ in asymmetric copulas. Moreover, the asymmetric copula can simulate the bivariate data in different ways even for the same dependence measure. As can be seen in Fig. 4 (e) and (f), the scatterings in these two are quite different even if they possess the same value of Spearman's ρ_s and even the same measure of asymmetry, e.g. η_∞. In Fig. 4 (e), it has higher lower-upper tail dependences than upper-lower tail dependences, e.g. $\lambda^{l,u}>\lambda^{u,l}$. This is different from Fig.

4 (f) which has a higher upper-lower dependences, e.g. $\lambda^{l,u} < \lambda^{u,l}$.



(a) Gaussian (b) Gumbel (c) Clayton



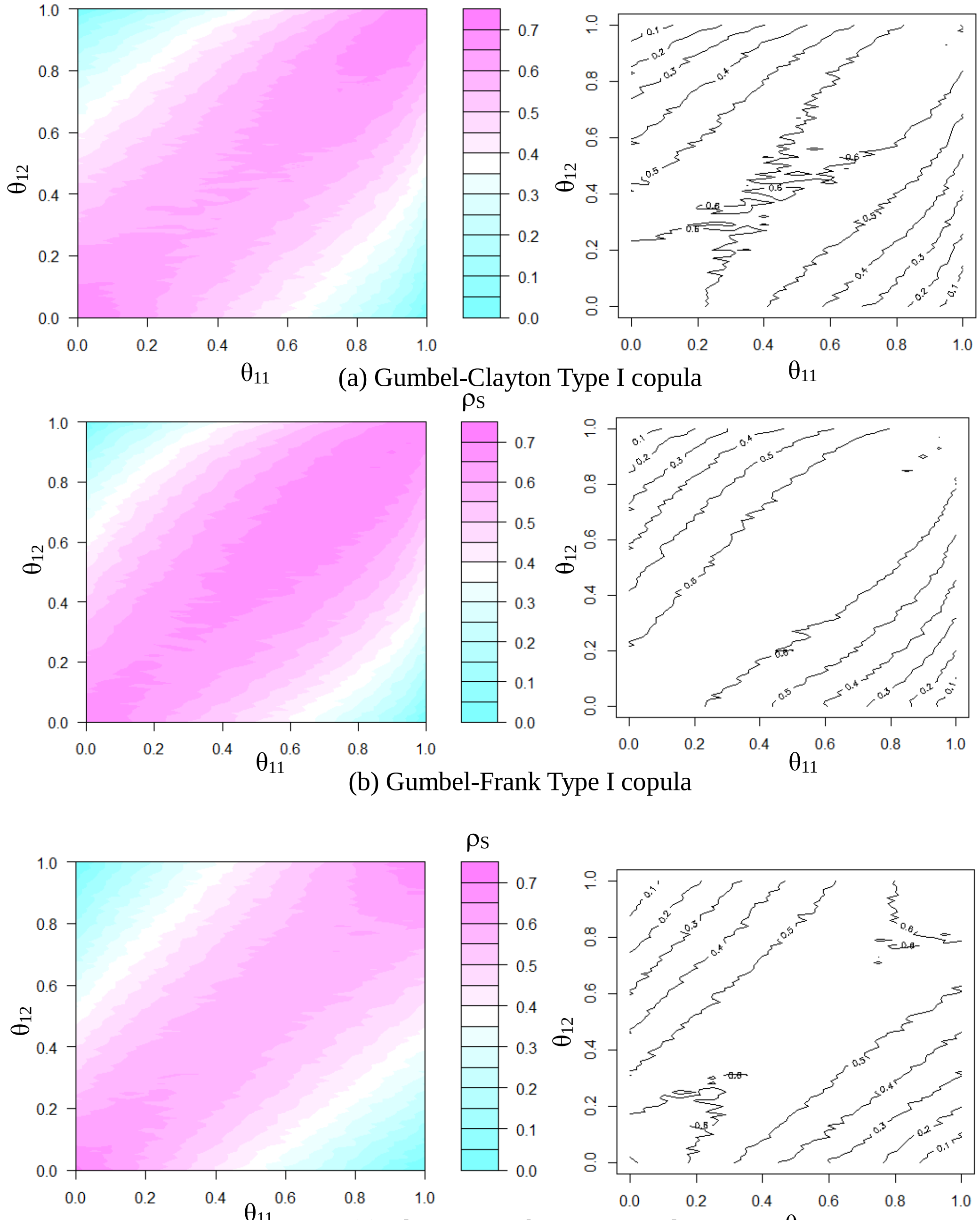
(d) Frank (e) Gumbel-Clayton (θ_{11}=0.3, θ_{12}=0.6) (f) Gumbel-Clayton (θ_{11}=0.6, θ_{12}=0.3)

Figure 4 Scatter plot of 5000 simulated samples from selected bivariate copulas

In fact, based on a given value of Spearman's ρ_s, the asymmetric copulas can characterize various types of dependences. Likewise, even if the base copulas are known, the asymmetric copula can still produce various values of Spearman's $\rho_s$by changing the parameter values in the individual functions. For example, by using the same base copulas, Gumbel, Clayton and Frank in Fig. 4, three asymmetric copulas can be formulated herein for a comparison. These are Gumbel-Clayton, Gumbel-Frank and Clayton-Frank asymmetric copulas and the type I individual function is used in the asymmetrizing. It should be expected that a change in θ_{11} and θ_{12}will result in changes in the dependence measures in each of these asymmetric copulas. Figure 5 illustrates the value changes of Spearman's ρ_s for the constructed asymmetric copulas when values of θ_{11} and θ_{12}change from 0 to 1.It can be seen the value of Spearman's $\rho_s$can change from the maximum 0.7 to the minimum 0 in all of the asymmetric copulas. In fact, when θ_{11}=θ_{12}, the value of Spearman's ρ_s is almost at the maximum, and when θ_{11}=1 and θ_{12}=0 or θ_{11}=0 and θ_{12}=1, the dependences can almost be neglected. A high similarity between the values of θ_{11} and θ_{12} would indicate a strong dependence while a small similarity implies independence. The values of θ_{11} and θ_{12} in the asymmetric copulas play a significant role in allocating the probability density concentrations in the copula domain. In other words, it can be realized the introduction of individual functions has added much more degrees of freedom in the dependence modeling of a copula function.

The existence of such asymmetric dependencies in the multivariate modeling should be paid attention to. A reliable multivariate model should be accurate enough in characterizing all the statistical properties of the dataset. In constructing the asymmetric copula model, the adjusting factors (e.g. the four parameters) could be estimated in such a way that both the linear dependences and tail dependences are well fitted. In other words, besides the statistics of goodness-of-fit, the tail dependence coefficients also need to be considered in assessing the quality of a copula model. For example, the maximum likelihood method and inverse Kendall's tau method could be applied to estimate the parameter. However, as discussed previously, when the information on the dependences of the data is only limited to correlations or covariance, plenty of copulas that possess such information can be employed. The use of one copula may not be able to depict the dependences very well. In reliability analysis of geotechnical problems, the influences of such subjective uncertainty in selecting either symmetric copulas or asymmetric copulas to the estimate of failure probability is still unknown. Since there exist such asymmetric dependences, the consideration of its influences to the reliability assessment should not be ignored. It is natural to question whether an asymmetric copula will produce significant different failure probabilities in the reliability problems when compared to a symmetric copula. Therefore, with this concern in mind, two examples are investigated in the next section.

ρ_S



(a) Gumbel-Clayton Type I copula

(b) Gumbel-Frank Type I copula

(c) Clayton-Frank Type I copula

Figure 5 Contour plot of the value of Spearman's ρ_s by changing the values of θ_{11} and θ_{12} in example asymmetric copulas

4. Case Studies

In this section, two geotechnical examples are studied to illustrate the impact of asymmetric dependences on reliability analysis; a continuous spread footing; and an infinite slope. The information of some soil properties in this study is considered to be partially known. Both the symmetric and asymmetric copula models are brought into the use of characterizing the dependences among soil parameters.

4.1 Example 1 - Continuous spread footing

The first example corresponds to a common strip foundation on the granite residual soil. The characteristics of the problem are presented in Fig. 6. The foundation is located below the ground with a depth of D meters and the width of the foundation is B meters. This fill soil has a unit weight of 17.5 kN/m^3 whereas the soil below the footing presents different mechanical and index properties. The loads were assumed to have a characteristic values of 450 kN/m, for the permanent load G, and 100 kN/m for the variable load Q. The foundation was designed in accordance to Eurocode 7 (EC 7) (Frank, 2004).

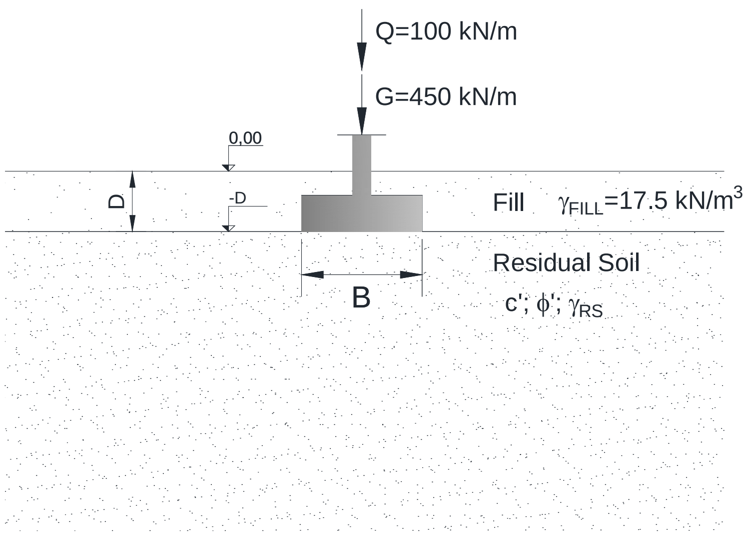


Figure 6 Strip Foundation for the worked example

The properties of the residual soil below the footing are the ones defined in the paper Zhang et al., (2018b). In this paper, the soil was extensively characterized and several distributions were fitted to the data, allowing the definition of the best distribution. The detailed information of measured data for the cohesion, c'_p, the peak friction angle, ϕ'_p. and the soil unit weight γ are presented in Table 2. By using the Akaike Information Criterion (AIC), the best marginal distributions are identified for each soil parameters as recorded in Table 3. There is no evidence showing the unit weight has dependences on cohesion and friction angle as indicated in Table 4. Thus, only the dependence between cohesion, c'_p and the peak friction angle ϕ'_p. are considered in the soil data multivariate modeling. Based on the equations provided in Section A.2, the measure of asymmetry is estimated for (c'_p, ϕ'_p). The results are recorded in Table 5. The measure of asymmetry has non-zero value and the upper-lower tail dependence coefficient is not the same as the lower-upper tail dependence coefficient. This indicates the bivariate data (c'_p, ϕ'_p) has asymmetries in its dependences.

Table 2Measured soil cohesion, friction angle and unit weight (Zhang et al. 2018b)

c'$_p$ (kPa)	**tan(ϕ_p')**	**γ (kN/m^3)**	**c'$_p$ (kPa)**	**tan(ϕ_p')**	**γ (kN/m^3)**	**c'$_p$ (kPa)**	**tan(ϕ_p')**	**γ (kN/m^3)**
11.68	0.85	19.19	53.75	0.37	19.23	1.22	1.01	18.96
10.91	0.87	19.41	10.03	0.75	19.87	30.38	0.68	19.2
12.04	0.86	19.44	10.95	0.76	19.07	0	1.19	19.02
36.69	0.58	19.52	1.4	0.81	17.78	2.98	0.77	19.11
0	1.19	18.29	51.12	0.25	18.35	5.23	0.85	18.78
13.79	0.73	19.03	10.89	0.75	19.4	33.74	0.53	19.22
13.84	0.82	20.27	1.96	1.02	19.4	18.65	0.56	18.86
47.85	0.45	19.1	0	1.19	19.57	5.16	0.94	18.7
5.62	1.05	17.23	34.14	0.6	19.43	8.6	0.86	18.14
18.93	0.68	19.19	5	1.01	19.06	22.79	0.7	19.81
14.61	0.76	17.72	16.23	0.69	18.5			
55	0.32	19.29	14.04	0.76	19.58			
6.44	1	19.27	48.22	0.38	18.02			
12.34	0.85	19.2	2.36	0.96	19.3			
5.56	0.91	18.87	0.17	1.02	17.46			

Table 3 Calculated AIC statistics for the marginal distribution model fitting

	Weibull	Normal	Lognormal	Logistic	Extreme value	Exponential	Gamma
c'_p(kPa)	299.8	340.2	329.2	339.1	356.3	303.5	295.2*
tan(ϕ_p')	-0.8646*	1.274	11.342	2.456	-0.3456	61.02	6.654
γ(kN/m^3)	51.36*	52.12	55.62	54.92	53.74	319.4	54.4

*The lowest AIC indicates the best model.

Table 4Dependences among soil parameters

	(c', tan(ϕ_p'))	(c',γ)	(tan(ϕ_p'),γ)
Correlation coefficient	-0.91	0.11	-0.09

Table 5 Measure of asymmetric dependences

	Measure of asymmetry η_∞	Lower-Upper Tail Dependence Coefficient at u=0.4	Upper-Lower Tail Dependence Coefficient at u=0.4
(c', tan(ϕ_p'))	0.011	0.03	0.05

Obviously, the sample size is a bit small for determining the exact joint distributions of the soil parameters. In any case, in geotechnical practices, either the information is scarce and no real statistics are possible, or simple statistics are applied. In such conditions, the full information of the residual soil properties is merely known. Therefore, the constructed multivariate models for these soil parameters need to take care of the uncertainties resulted from data scarceness. In that sense, it should be realized various joint models can be possibly applied in fitting the multivariate soil data. Thus, the following analysis will consider the uncertainties associated with the model selections.

Since no clear copulas have been specified for the dependences between cohesion and friction angle, several asymmetric copulas, as introduced in Section 3, are utilized here to model the soil data for the given correlation coefficient. To compare with the symmetric copula, the commonly adopted symmetric Archimedean copulas are also considered in this modeling of dependences. However, as there are many combination rulesin constructing the asymmetric copulas, it is impossible to investigate all types of asymmetric copulas. Thus, in order to make the problem simpler, this study will only utilizes the commonly adopted Archimedean copulas as the base copulas for the construction of asymmetric copulas. The most commonly applied Archimedean copulas that can characterize different tail dependences are used in this study, namely, Gumbel, Clayton and Frank copulas. Based on the construction rules, the asymmetric copulas are established based on these selected base copulas. Specifically, the following types of copulas are investigated and compared in modeling the cohesion and friction angle with the same given correlation coefficient:

1. *Gaussian copulas*: The most widely applied Gaussian copula is applied herein. The Gaussian structure is considered to represent the dependences in the copula domain.

2. *Symmetric copulas*: The classic symmetric one parameter Archimedean copulas are considered in the modeling. These are the most famous families, which features a wide range of tail dependences, namely Gumbel, Clayton and Frank copulas.

3. *Type I asymmetric copulas*: We adopt the Khoudraji's device for the construction of asymmetric copulas. Based on Eq. (1), we combine two base copulas from the selected Archimedean copulas. This produces three combinations namely, Gumbel-Clayton Type I, Gumbel-Frank Type I and Clayton-Frank Type I asymmetric copulas. For the individual functions, the Type I function listed in Table 1 is selected for the asymmetric copula construction.

Meanwhile, it should be realized the Gumbel, Clayton and Frank copulas are usually used to characterize positive dependences. For the current case, as cohesion and friction angle are negatively dependent, a direct use of these copulas to the data will have problems in parameter estimations. Therefore, for the ease of modeling, a simple modification in the data can be applied. Instead of directly modeling the original data, the copula models are utilized to model the (-c'_p, tan(ϕ'_p)) instead of (c'_p, tan(ϕ'_p)). Since copula only cares about variables' cumulative distribution function values, such change will have no influence on the quality of copula model. The marginal distribution models for the soil variables will remain unchanged.

The results for the log-likelihood and AIC statistics for all the considered models fitting to (-c'_p, tan(ϕ'_p)), are presented in Table 6. The total log-likelihood refers to the summation of log-likelihood from both marginal distribution functions and copula function. As shown in the results, Gumbel-Clayton Type I has the lowest AIC compared to the rest models. However, the AIC values of all these candidate copula models are quite close. In fact, the goodness-of-fit test shows that all the candidate copula models could be used in the fitting to the bivariate data without rejections. Therefore, in the following, all these models will be used in the analysis and compared with each other. As there is no clear judgment in the model selections, we would like to accept them all. However, the analysis will be focusing on the differences in the reliability estimates which are resulted from using different copulas.

Table 6 Comparison of copula parameter estimates and AIC statistics to the data of (c'_p, tan(ϕ'_p))

Copula type	Total log-likelihood	No. of parameters	AIC
Gaussian	35.53	5	-61.06
Gumbel	37.81	5	-65.62
Clayton	34.81	5	-59.62
Frank	34.17	5	-58.34
Gumbel-Clayton Type I	41.45	8	-66.9*
Gumbel-Frank Type I	41.20	8	-66.4
Frank-Clayton Type I	40.48	8	-64.96

*Minimum AIC value indicates the best model.

In our example, as the idea is to use the analytical expression for the load capacity of the foundation, hence a unique value for the deterministic parameters (e.g. G, Q, D and γ_{Fill}) is used in each calculation. The foundation will be designed according to the analytical formula given in EC 7. The bearing capacity of the foundation is defined as:

$$q_{ult} = c' \cdot N_c + q' \cdot N_q + \frac{1}{2} \times \gamma^* \cdot B' \cdot N_\gamma \tag{2}$$

where the terms N_c,N_q and N_γ are the capacity factors, depending only on the friction angle of the ground and defined by the following expressions (Bond et al., 2016):

$$N_q = e^{\pi \cdot tan\phi\prime} \cdot tan^2\left(45° + \frac{\phi'}{2}\right) \tag{3}$$

$$N_c = \left(N_q - 1\right) \cdot cot\phi' \tag{4}$$

$$N_\gamma = e^{\frac{1}{6}*\left(\pi + 3\pi^2 tan\phi\prime\right)} \times \left(tan\phi'\right)^{2\pi/5} \tag{5}$$

The term q' corresponds to the effective stress at the base of the foundation which, in the present case, is:

$$q' = \mathrm{D} \times \gamma_{Fill} \tag{6}$$

D is the depth of the footing and γ^*corresponds to average submerse unit weight of the ground below the foundation level and, in the present case, as the water level is not considered, is equal to the unit weight of the residual soil γ. $B'$is the effective width of the foundation being equal in the present case to *B* as only vertical loads are acting on the foundation. In such conditions, the ultimate vertical load strength of the foundation is equal to:

$$Q_{ult} = q_{ult} \times B \tag{7}$$

And, according to EC7, the following inequality should be satisfied in order to verify the ultimate limit state for bearing resistance:

$$V_d \leq R_d \tag{8}$$

where V_d is the vertical variable load and R_d is the bearing resistance. Applying the partial Factors of Safety proposed by the Eurocode 7, a value of safety factor 1.25 is assigned to the cohesion and $tan(\phi')$, and 1.3 to the variable loads, and thus a dimension *B*=2.5 m satisfies the safety requirements proposed by Eurocode 7.

The following step consisted in the evaluation of the Safety Margin, given the foundation geometry, namely *B*=2.5 m and *D*=1.0 m. For this purpose no partial Factors of Safety are applied and thus:

$$G + Q \leq Q_{ult} \tag{9}$$

where *Q* and *G* refer to the dead and live loads transferred to the shallow foundation. Thus the safety Margin (M) can be defined as:

$$M = Q_{ult} - G - Q \tag{10}$$

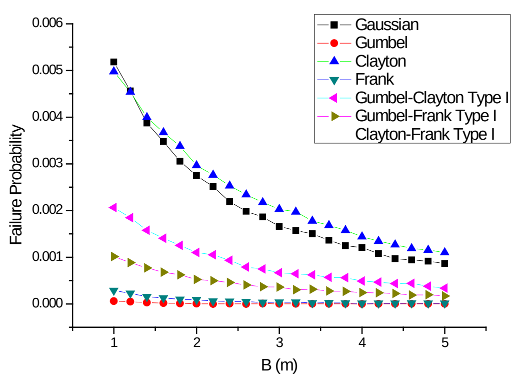
As for the residual soil studied here, Monte Carlo simulations with 10000 samples are used in the computation for representing their randomness. The associated copulas are utilized in the dependence modeling separately. The computed results for the failure probabilities and factor of safety is shown in Table 7. It can be seen the failure probabilities differs quite a lot among the copulas. The highest failure probability is $2.03 \cdot 10^{-3}$ in Frank copula and the lowest failure probability is $2.00 \cdot 10^{-6}$ in Clayton copula. Although the computed failure probabilities differ a lot, the factor of safety does not show very large variations over different copulas. The main reason is because the failure probabilities are often related to distribution tails while the factor of safety is a measure of distance from the performance function mean to the safety margin. Therefore, even the value of factor of safety is very close, it could not simply imply a similar value in the failure probability. The dependences have great influences in the safety assessment.

Table 7 Computed failure probabilities and safety factor for the initial value in footing example.

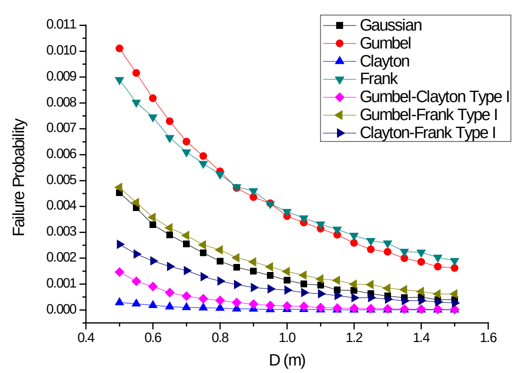
	Gaussian	Gumbel	Clayton	Frank	Gumbel-Clayton Type I	Gumbel-Frank Type I	Clayton-Frank Type I
Failure probability	$4.19 \cdot 10^{-4}$	$1.66 \cdot 10^{-3}$	$2.00 \cdot 10^{-6}$	$2.03 \cdot 10^{-3}$	$3.70 \cdot 10^{-5}$	$6.67 \cdot 10^{-4}$	$3.58 \cdot 10^{-4}$
Factor of safety	8.9616	9.0208	8.9571	8.9725	9.0207	8.9400	8.9788

To further explore the influence of asymmetric dependences on the reliability analysis, the following four factors are systematically studied: (1) the width *B* of the foundation; (2) the depth *D* of the foundation; (3) mean value of residual soil unit weight and (4) correlation coefficient between cohesion and friction angle. These investigated factors are in fact corresponding to engineering and research concerns. The width and depth of the spread footing are the primary concern from the design perspective. The mean value of the residual soil is associated with the uncertainties of geological materials and measurement. The study of the correlation coefficient is referred to the consideration of influence of dependences. Thus, in this parametric study, the failure probabilities and the factor of safety are both computed in each cases when each factor is varied over a range of values.

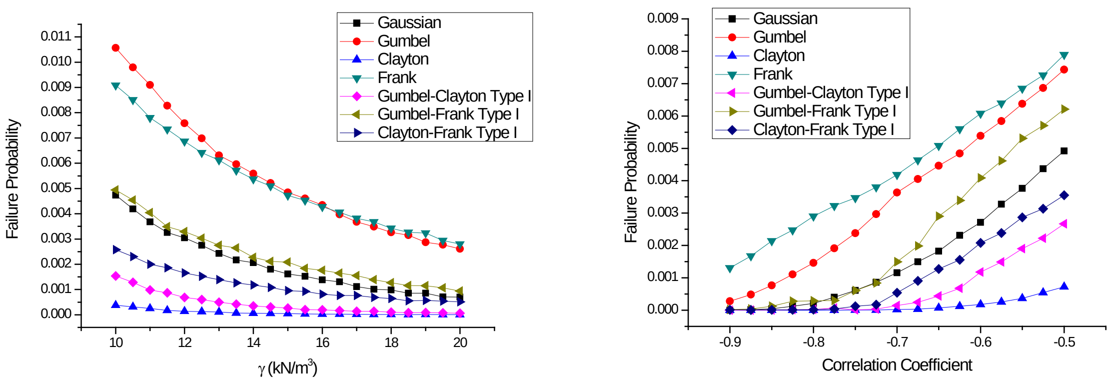
Figure 7(a) shows the computed failure probabilities for the spread footing when *B* changes from 1 m to 5 m. It is observed the symmetric copulas, Frank and Gumbel, produce the largest failure probabilities for all the considered *B* values and the Clayton copula produces the lowest failure probabilities. Among the asymmetric copulas, the Gumbel-Frank Type I copula produces the largest failure probabilities whereas Gumbel-Clayton Type I produces the lowest probabilities. The Gaussian copula produces a moderate value of failure probabilities which is in between the highest and lowest. These results imply that the differences in probabilities of failure produced by symmetric and asymmetric copulas are quite significant. The same conclusions can be drawn from Fig. 7(b), which shows the variations in failure probabilities regarding the change of *D*. The results are quite similar to the case in Fig. 7(a) despite the sensitivity of the failure probabilities. It is seen the failure probabilities changed from 0.0101 to 0.0016 when *D* changes from 0.5 m to 1.5 m by adopting the Gumbel copula. This is much larger compared to the change of *B* from 1 m to 5 m. For this particular case, it indicates the foundation depth *D* is more important than the width *B* in the reliability assessment.



(a) *B* changes from 1 m to 5 m



(b) *D* changes from 0.5 m to 1.5 m



(c) Mean of γ changes from 10 kN/m³ to 20 kN/m³ (d) Correlation changes from -0.5 to -0.9

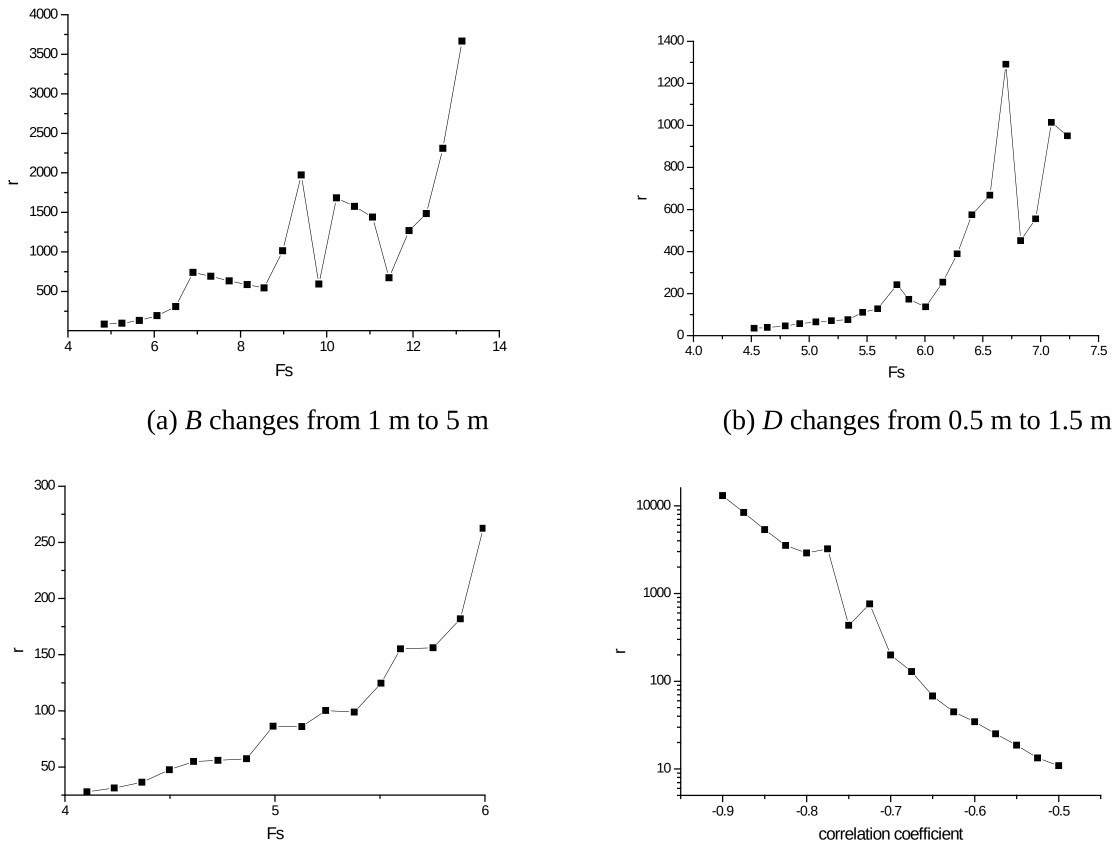
Figure 7 Probabilities of failure for the spread footing by using different copula models

Compared to the geometric factors, the influence of mean value of soil unit weight to the failure probabilities is even more critical. As shown in Fig. 7(c), the largest failure probability is 0.0105 and lowest failure probability is $6.00 \cdot 10^{-6}$ when mean of γ changes from 10 kN/m³ to 20 kN/m³. Obviously, the value of this soil parameter in real nature might not have such a wide variation. Here, the analysis of the parameter value is for the purpose of parametric understanding. The investigated range of parameter values is selected arbitrarily. The largest failure probabilities are produced from Gumbel and Frank copulas while the lowest failure probabilities are produced from Clayton copula. Again, the failure probabilities produced by asymmetric copulas are bounded by the symmetric copulas. The same observation can be obtained by looking at the influences of correlations between the soil parameters to the failure probabilities in Fig. 7(d). The failure probability increases as the correlation coefficient increases. The largest failure probability is produced from Gumbel copula while Clayton copula produced almost all the smallest failure probability. The performance of the asymmetric copulas is quite the same as the other cases in Fig. 7.

In order to show the maximum possible dispersion in the failure probability of the problem when dependence structures varies within the set of both symmetric and asymmetric copulas, a global dispersion factor associates with the failure probability is utilized here (Tang et al., 2015). This is defined as following

$$r = \frac{p_{f,max}(C)}{p_{f,min}(C)} \tag{11}$$

where$p_{f,min}(C) = \min\{p_f(C), C \in \Theta\}$ and $p_{f,max}(C) = \max\{p_f(C), C \in \Theta\}$ in which $p_f(C)$ is the failure probability of the spread footing associated with a specific copula *C*. The set of copulas Θ would include all the considered symmetric and asymmetric copulas, e.g. $\Theta = \{\text{Gaussian, Gumbel, Clayton, Frank, Gumbel} - \text{Clayton Type I, Gumbel} - \text{Frank Type I, Clayton} - \text{Frank Type I}\}$.



(a) *B* changes from 1 m to 5 m

(b) *D* changes from 0.5 m to 1.5 m

(c) Mean of γ changes from 10 kN/m^3 to 20 kN/m^3

(d) Correlation changes from -0.5 to -0.9

Figure 8 Dispersion factor of the probabilities of failure for the spread footing

The results of the calculated dispersion factor are shown in Fig. 8. In order to make a fair comparison, the factor of safety is also calculated and set as the horizontal axis for all the soil parameters. For the geometric factors, the increase of *B* and *D* both lead to an increase of *r* although some fluctuations exist in the trend. The increase of the mean of soil unit weight lead to an obvious increase in the dispersion factor. However, the increase of correlation coefficient results in a decrease of *r*. All the results showed that the dispersion factor becomes quite large when failure probability is small. The differences in the failure probabilities can be several orders of magnitudes. This generally implies the ignorance of the dependences would be quite problematic when estimating small failure probabilities.

4.2 Example 2 - Infinite slope

The second example considers the reliability analysis of an infinite slope with consideration of soil parameter uncertainties. This example corresponds to an infinite slope in a residual soil. The uncertainties regarding the soil properties including cohesion, friction angle and soil unit weight are again considered in this case study. The copula models as constructed in Section 4.1 are used again to characterize the cohesion and friction angle. The results of this example are used to compare with the above example in order to see whether the asymmetric dependences will still have large impact on the reliability results when performance function changes. The investigated slope is represented in Fig. 9, with the parameters of the residual soil also presented. In this example, the water content in the soil is not considered.

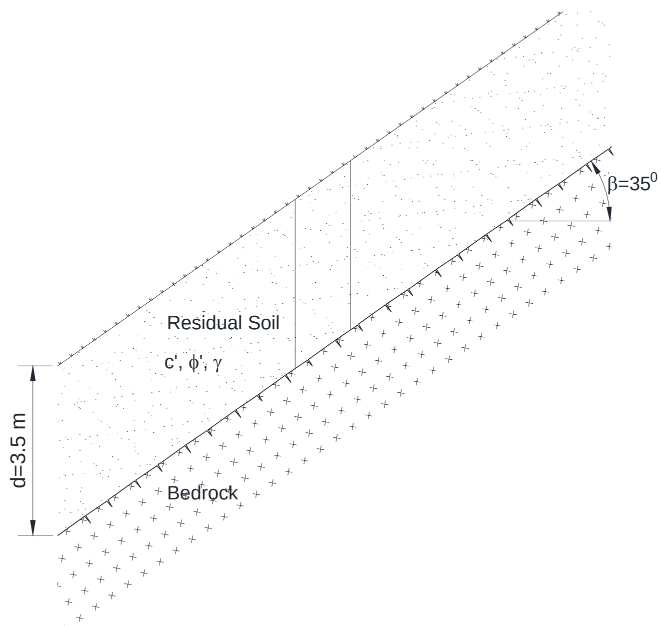


Figure 9 Infinite slope on residual soil

For infinite slopes the equilibrium can be established from a unitary width slice which can be shown in Fig.10:

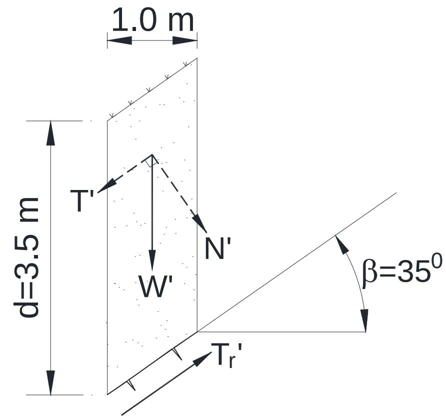


Figure 10 Equilibrium for a unitary width slice

For this soil slice, the equations can be established as follows

$$W' = \gamma \times d \tag{12}$$

$$N' = W' \times cos\beta \tag{13}$$

$$T' = W' \times sin\beta \tag{14}$$

$$T_r = c' \times \frac{1}{cos\beta} + N' \times tan\phi' \tag{15}$$

where γ is the soil unit weight, d is the depth of soil slice and β is the angle of the slope. Therefore, the reliability of the infinite slope can be evaluated by the safety margin given by

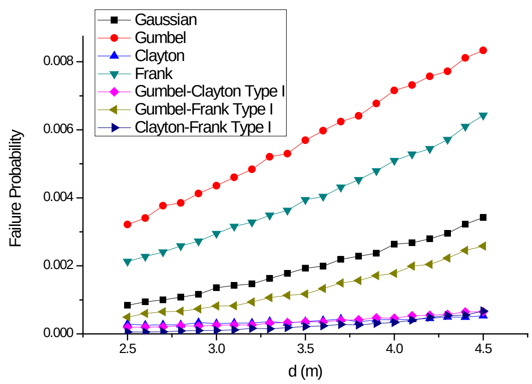
$$M = T_r - T' \tag{16}$$

The associated performance function would be same as Eq. (10) while a value of M less than 0 is believed to be a failure in the infinite slope.

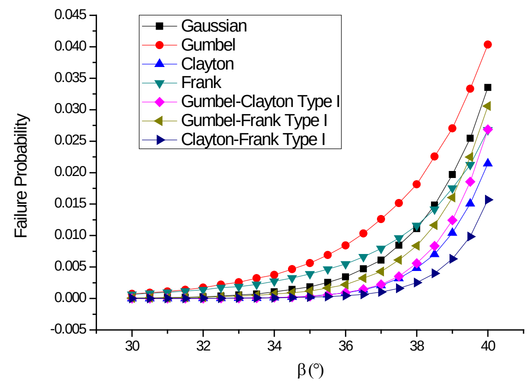
The same calculation procedures are repeated for this infinite slope problem. As an initial case, the factors of the slope geometry are set at d=3.5 m and β=35°. The properties of the soil are considered the same as for the footing example. The calculated failure probabilities and factor of safety are recorded in Table 8. Compared to the spread footing, the differences in the failure probabilities using different copulas become smaller. The largest failure probability is $5.69 \cdot 10^{-3}$from the Gumbel copula and lowest failure probability is $2.13 \cdot 10^{-4}$from Clayton-Frank Type I copula. The computed safety factors are much smaller compared to that in the previous example.

Table 8 Computed failure probabilities and safety factor for the initial value in infinite slope example.

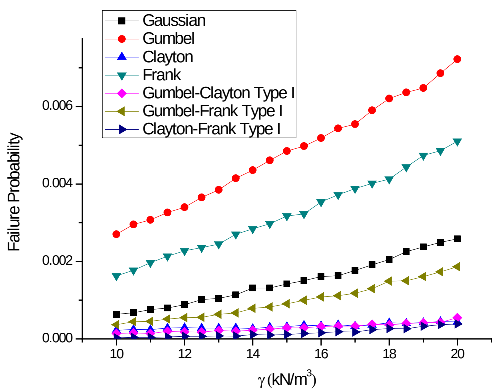
	Gaussian	Gumbel	Clayton	Frank	Gumbel-Clayton Type I	Gumbel-Frank Type I	Clayton-Frank Type I
Failure probability	$1.93\cdot10^{-3}$	$5.69\cdot10^{-3}$	$3.63\cdot10^{-4}$	$3.94\cdot10^{-3}$	$3.56\cdot10^{-4}$	$1.17\cdot10^{-3}$	$2.13\cdot10^{-4}$
Factor of safety	1.5211	1.5210	1.5215	1.5207	1.5213	1.5212	1.5217



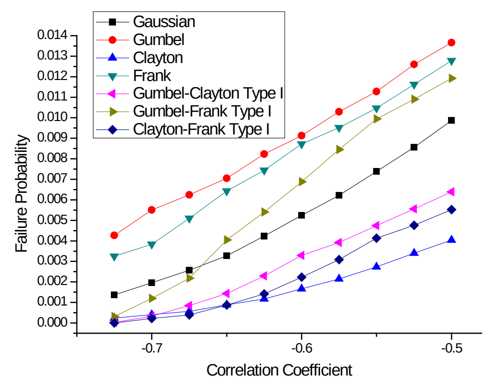
(a) d changes from 2.5 m to 4.5 m



(b) β changes from 30° to 40°



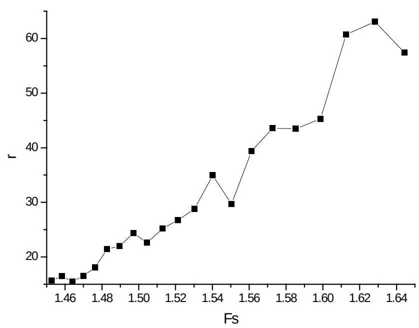
(c) Mean of γ changes from 10 kN/m^3 to 20 kN/m^3

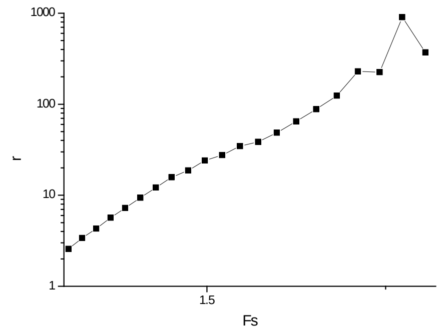


(d) Correlation changes from -0.5 to -0.725

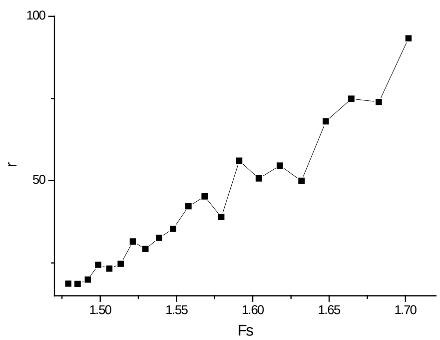
Figure 11 Probabilities of failure for the infinite slope by using different copula models

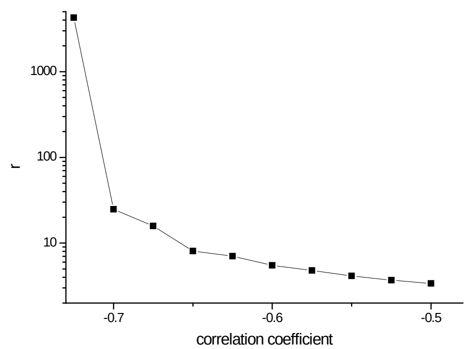
In this example, we consider the following parametric studies: (1) the depth d of the soil slice; (2) the angle β of the slope; (3) mean value of the slope soil unit weight and (4) correlation coefficient between cohesion and friction angle. Again, these investigated factors are related to the engineering concerns on the analysis of slope stability. Following the same way of computations, the failure probabilities and safety factors for the slope are computed in each case when each factor is varied over a range of values. The results are plotted in Fig. 11. The influence of the dependences to the reliability analysis is also presented by the dispersion factors. By using the same formula as the previous example, the dispersion factors for these four parameters are computed and plotted in Fig. 12.





(a) d changes from 2.5 m to 4.5 m (b) β changes from 30° to 40°





(c) Mean of γ changes from 10 kN/m^3 to 20 kN/m^3 (d) Correlation changes from -0.5 to -0.725

Figure 12 Dispersion factor of the probabilities of failure for the infinite slope

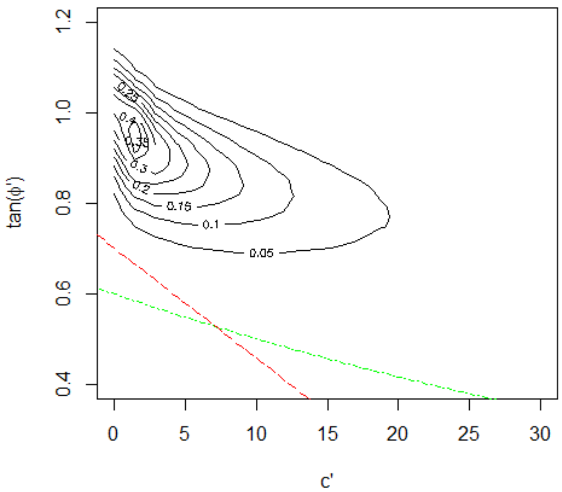
It can be seen the computed failure probability differs considerably. The failure probability is very sensitive to the type of copulas. Meanwhile, all the dispersion factors increase with the decrease of failure probability which is quite similar as the footing example. It is observed the influence of dependences to the failure probability is also very significant in this example. The value of dispersion factor can go up to a magnitude of 10^4. It showed again that the failure probability is very sensitive to the dependences between the soil variables.

4.3 Discussion of the Results

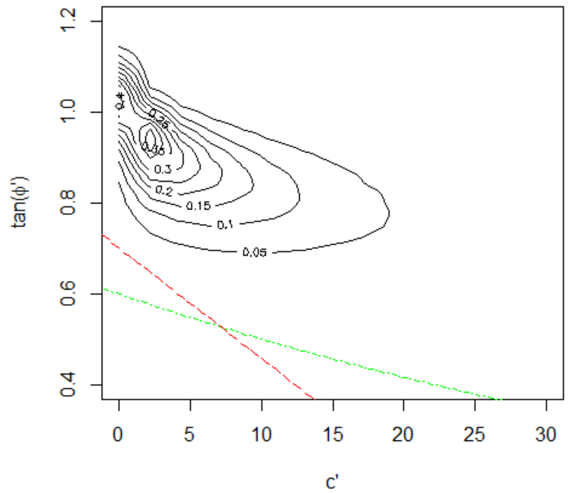
Based on the above results, it can be concluded that the failure probability of spread footing and the failure probability of infinite slope associated with different dependences differ greatly, especially for estimating small failure probabilities. Asymmetric copulas in these cases also showed a big difference from symmetric copulas. To provide a better explanation of the differences in failure probabilities, a comparison among the joint probability density function isolines of cohesion and friction angle is made for all the copulas. These contour lines and the limit states for the spread footing and infinite slope are both plotted in Fig. 13. The limit states for the considered problems as plotted here have adopted a deterministic value for the soil unit weight. A key different between the asymmetric copulas and symmetric copulas is the tail dependences. Compared to the symmetric copulas (Gaussian, Gumbel, Clayton and Frank), the asymmetric copulas have a small *lower-lower* tail dependences. This can be indicated by the contour lines where the symmetric copulas have a much wider area compared to asymmetric copulas. It is also observed from the contour plot that the *lower-lower* tail of asymmetric copulas is not symmetric. This also means the estimation of high quantile in a copula model might be different when using an asymmetric copula in the dependence modeling.

The limit states of the considered geotechnical problems are almost lying in the *lower-lower* region. That is why the Gumbel copula always produces the largest failure probability as dependences in Gumbel copula are concentrated at the *lower-lower* region. For the rest copulas, the probability densities are not only concentrated at the *lower-lower* region, therefore, the failure probability is quite small. However, this phenomenon may not be true for other problems. For example, if the limit state is lying in the *lower-upper* region or *upper-lower* region, the asymmetric copulas may produce the maximum or minimum failure probabilities. The consideration of asymmetric dependences in the reliability is indeed a necessary factor. Meanwhile, it should be noticed the provided data sample in this study is quite limited. This is also the reason why the copula models cannot be easily identified. However, this is quite common in engineering applications as data scarceness problems can be frequently met in real practices. When copula function is not easily identified, the information of dependences will also be hardly captured. With only limited information about the relationship among the random variables, asymmetric copula may produce a result which can differ significantly from the symmetric copula. A further comparison is also provided for the investigated examples with consideration of different degrees of asymmetric dependences. Here, the adopted best asymmetric copula as highlighted in Table 6 is utilized, i.e. Gumbel-Clayton Type I. However, the weighting parameters θ (as given in Table 1) in this copula are adjusted to obtain asymmetric copulas having different measure of asymmetry, i.e. η_∞=0, η_∞=0.001 and η_∞=0.01. The reliability analysis is repeated for each of these asymmetric copulas. The results are recorded in Table 9. It can be seen in both examples as the degree of asymmetric dependences increases, the failure probability increases. This agrees well with the aforementioned finding that since the performance function is lying in *lower-upper* region, the asymmetric copulas may produce larger failure probabilities compared to symmetric ones.

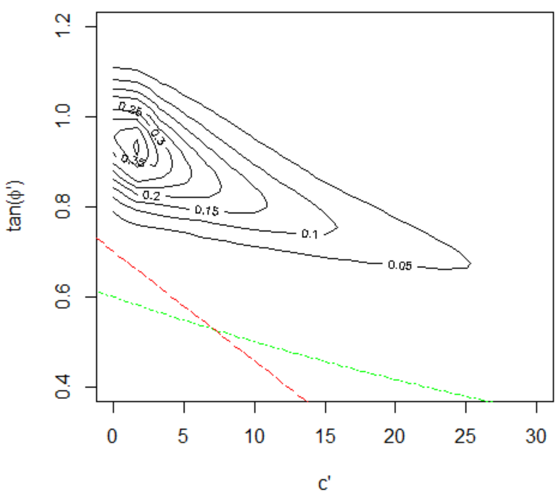
The investigation shows that the asymmetric copula approach provides another alternative way in the modelling and processing of dependent variables. This asymmetric copula approach has demonstrated to be able to produce different results in the reliability analysis compared to the symmetric copula approach. On the one hand, the room for indeterminacy in dependence models reduces the risks of too optimistic conclusions, which could be made from a traditional symmetric copula approach under rough assumptions. On the other hand, the characterization of asymmetric dependences provides a much more flexible way of modeling the real observations. In view of making critical engineering decisions, direct emphasis can be put on the extreme values in the estimated bounds for the failure probability. In this manner, global sensitivities such as failure probability with respect to dependence modeling can be revealed. Conversely, optimal design can be directly made based on specified constraints for the results such as allowable largest failure probability values. Such model can take into account various dependences including symmetric and asymmetric dependences associated with the geotechnical problem in a quantitative manner.



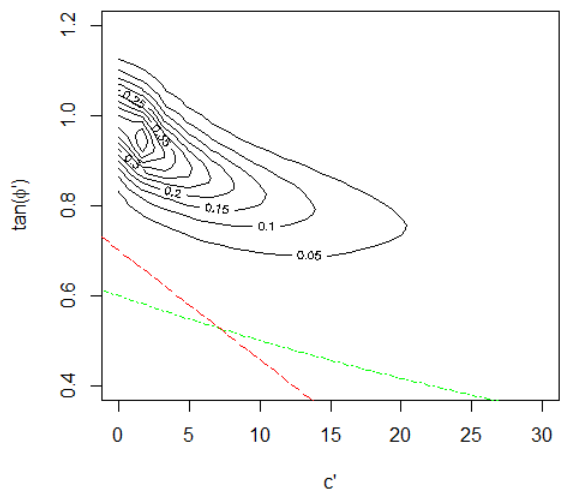
(a) Gaussian



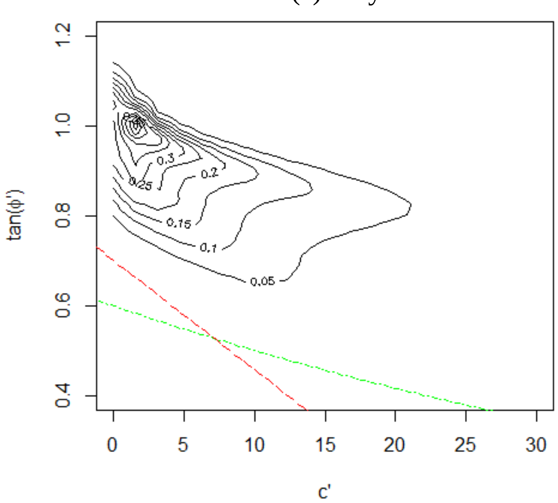
(b) Gumbel



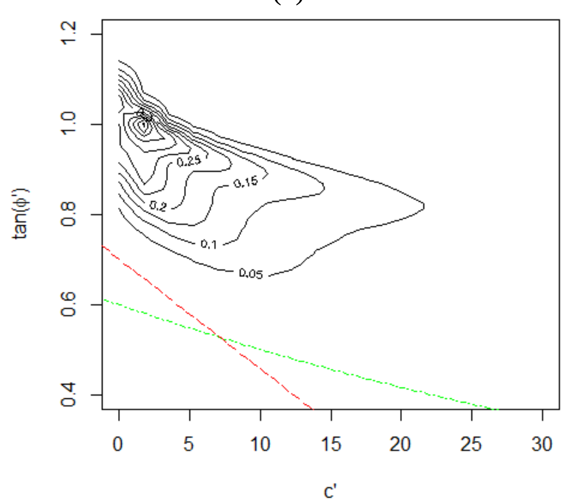
(c) Clayton



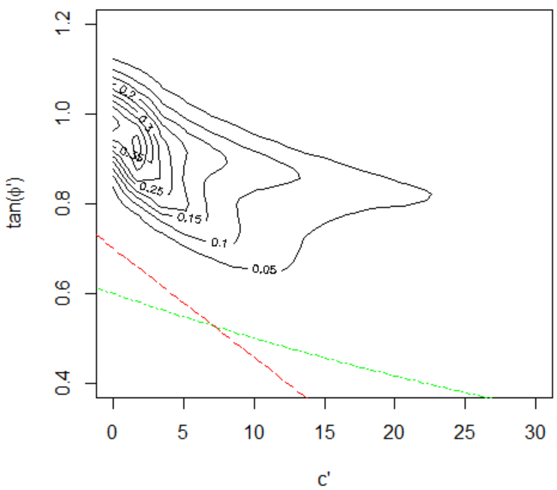
(d) Frank



(e) Gumbel-Clayton Type I



(f) Gumbel-Frank Type I



(g) Clayton-Frank Type I

Figure 13 Contour plots of different copulas and the limit states in two geotechnical examples (red dashed line represents the spread footing limit state with γ=18.9 kN/m^3, green dotted line represents the infinite slope limit state with γ=18.9 kN/m^3)

Table 9 Comparison of failure probabilities for different degrees of asymmetric dependences.

	η_∞=0	η_∞=0.001	η_∞=0.01
Example 1	$2.23 \cdot 10^{-6}$	$7.52 \cdot 10^{-6}$	$1.95 \cdot 10^{-5}$
Example 2	$1.09 \cdot 10^{-5}$	$5.87 \cdot 10^{-5}$	$2.16 \cdot 10^{-4}$

5. Conclusions

In this paper, the influence of asymmetric dependences to the reliability analysis has been analyzed by means of the asymmetric copulas in a multivariate setting. The fundamental methodology including the asymmetrizing techniques in formulating an asymmetric copula is introduced in detail, which includes the theoretical concepts of measuring the asymmetric dependences and tail dependences for a copula model. Geotechnical engineering problem is utilized in this study for the investigation of the influences of asymmetric dependence to the reliability analysis. Based on selected Archimedean copulas, the asymmetric copulas were constructed and then compared with traditional symmetric copulas on the modeling of soil parameters for the reliability analysis of a spread footing and an infinite slope. The results showed the computed failure probabilities and factors of safety differ significantly among the selected copulas.

Although one expects different results to be produced by symmetric copulas and asymmetric copulas, the magnitude and significance of these differences have not been reported. It was shown the ignorance of the asymmetric dependence in the reliability analysis might create large errors in the results. It is of practical importance to select the most appropriate copula in characterizing the dependence structure of soil parameters. The ignorance of asymmetric dependences might largely reduce the accuracy in the reliability analysis or risk assessment when only limited information of variables is known. However, it should be pointed out the results obtained from the present study can only be interpreted for the investigated geotechnical examples. The parameter may exhibit different dependences in other situations when engineering problem changes. Moreover, it also should be realized the number of considered candidate asymmetric copulas is small. There are still many more asymmetric copulas that could be constructed from the procedures introduced in this paper. Thus, the results may also be distorted if other copulas are adopted. The conclusions drawn from the thesis should be seen in the light of these limitations. The influence of these limitations to the reliability results may need further investigations in the future. Future work seems necessary to investigate the ways of selecting base copulas and individual functions in the construction of asymmetric copulas. Also, applications of the obtained asymmetric copula to real engineering problems, as well as different performance functions, may prove to have relevant interest regarding Reliability Based Design.

Appendix A Fundamental knowledge of copulas and dependence concepts

In this section, the fundamental knowledge of copula as well as dependence/asymmetric dependence concepts are briefly introduced.

A.1 Definition and basic properties

In general, a copula is a model which couples a multivariate distribution to its one-dimensional marginal distributions. The fundamental definition of copula originates from the Sklar's theorem (Sklar, 1959):

Sklar's Theorem: Let H be a joint distribution function for n random variables with marginal distributions $H_1, \ldots, H_n$. A copula C is then defined as an n-dimensional joint distribution function such that for all x ∈

R^n

$$H\left(x_1,\ldots,x_n\right)=C\left(H_1\left(x_1\right),\ldots,H_n\left(x_n\right)\right) \tag{A.1}$$

If $H_1, \ldots, H_n$ are all continuous, then C should be unique. As seen in its formulation, the copula function does not need to cater about the marginal distribution of the random variables. This is because the integral transform which transforms random variables to their cumulative distribution function values $u_i=H_i(X_i)$ has turned all the random variables in a copula to be uniformly distributed variables within [0, 1]. Therefore, the domain and range of values for an n-dimensional copula function is

$$C:[0,1]^n \rightarrow [0,1].$$

The copula approach has the freedom of selecting any marginal distributions for the variables, which makes it much more flexible, compared to the traditional joint distribution models in characterizing individual variable's behaviors. Many well-known developed copula functions and families have been applied in various fields; see e.g. (Hutchinson and Lai 1990; Trivedi & Zimmer, 2007).The most commonly applied copulas are the Archimedean copulas which can be expanded to a high multivariate model through straightforward transformations (Genest & Rivest, 1993).

A.2 Dependence measures

When addressing the significance of the copula approach in modeling multivariate data, the concepts of dependence should be explained in detail herein. In measuring the dependence of multivariate data, the Pearson's correlation coefficient ρ is most commonly applied as it could depict the linear dependences among the data. Obviously, the concept is too simple and biased and, thus, many researchers tend to criticize it (Phoon & Ching, 2014). Generally, if the data shows a perfect linear relationship, e.g. ρ = 1, the dependency is well represented by the correlation coefficient. However, if the data is observed to be imperfect linearly dependent, e.g. $-1<\rho<1$, the value of the correlation coefficient could be questionable in measuring the dependence. Moreover, it is also known the linear correlation coefficient is very sensitive to the marginal distributions of the variables. As such, other concepts of dependencies have been brought into the use in measuring the dependences. The concepts such as Kendall's τ_k and Spearman's ρ_s are considered as more robust dependence measures. Kendall's τ_k is a measure of the concordance/discordance in the data sample, and Spearman's ρ_s is a measure of the rank correlations (see Salvadori et al. 2007). Since these two measures are concordant measures of rankings among the variables, they are believed to be more robust when compared to Pearson's correlation.

A.3 Measure of asymmetry and tail dependency of a copula model

Many definitions of symmetric dependence in a copula model are developed in the literature. Among these, the concept of "exchangeability" is commonly adopted as the fundamental measure of symmetry for the copula model. This can be defined as following. For a given copula $C(u_1,\dots,u_n)$, if

$$C\left(u_1,\dots,u_i,\dots,u_j,\dots,u_n\right)=C(u_1,\dots,u_j,\dots,u_i,\dots,u_n) \text{ is true for any pair } u_i,u_j \in \mathbf{I},$$

then it is believed the copula $C(u_1,\dots,u_n)$can be said to be symmetric (Genest and Nešlehová, 2013). Therefore, if the above condition is not met, the copula is considered as asymmetric. Based on this concept, the measure of asymmetry in a copula model can be estimated as following (Klement and Mesiar, 2006)

$$\eta_p\left(C\right)=\left\{\int_0^1\int_0^1\left|C\left(u_1,u_2\right)-C\left(u_2,u_1\right)\right|^p du_1du_2\right\}^{1/p} \qquad \text{(A.2)}$$

where p is a factor which can be set at any value greater than or equal to 1. For the convenience, usually the value of p is set to be infinity in the measure of asymmetry. This leads to a simplified formula as

$$\eta_\infty\left(C\right)=\sup_{\left(u_1,u_2\right)\in[0,1]^2}\left|C\left(u_1,u_2\right)-C\left(u_2,u_1\right)\right| \qquad \text{(A.3)}$$

A large value of this measure implies a strong asymmetric dependence in copula.

Other than the measure of asymmetry, the tail dependences could also be used to detect the asymmetric characteristics. Fundamentally, the tail dependence coefficients include four types, namely, *lower-lower*, *lower-upper*, *upper-lower*, *upper-upper* tail dependence coefficients. In the case of bivariate copula $C(u_1,u_2)$, the calculation of these tail dependence coefficients is given by(Nelsen 2006)

$$\lambda_{1|2}^{l,l}(C) = \lim_{u \to 0+} P\left(x_1 \le H_1^{-1}(u) \mid x_2 \le H_2^{-1}(u)\right) = \lim_{u \to 0+} \frac{C(u,u)}{u} \quad \text{(A.4)}$$

$$\lambda_{1|2}^{l,u}(C) = \lim_{u \to 0+} P\left(x_1 \ge H_1^{-1}(1-u) \mid x_2 \le H_2^{-1}(u)\right) = 1 - \lim_{u \to 0+} \frac{C(u,1-u)}{u} \quad \text{(A.5)}$$

$$\lambda_{1|2}^{u,l}(C) = \lim_{u \to 0+} P\left(x_1 \le H_1^{-1}(u) \mid x_2 \ge H_2^{-1}(1-u)\right) = 1 - \lim_{u \to 0+} \frac{C(1-u,u)}{u} \quad \text{(A.6)}$$

$$\lambda_{1|2}^{u,u}(C) = \lim_{u \to 0+} P\left(x_1 \ge H_1^{-1}(1-u) \mid x_2 \ge H_2^{-1}(1-u)\right) = 2 - \lim_{u \to 0+} \frac{1-C(1-u,1-u)}{u} \quad \text{(A.7)}$$

where$H_1^{-1}(.)$ and $H_2^{-1}(.)$ are the inverse cumulative distribution functions for x_1 and x_2. Obviously, from Eqs. (A.4)-(A.7) we can see these calculations provide measures of the tail dependence for the two variables in four different extremes.

Tail dependencies can provide useful information regarding the asymmetric dependences from the intrinsic information. The comparison of *lower-upper* and *upper-lower* tail coefficients can be utilized as a reference in assessing the asymmetry of a copula. For example, in a symmetric copula, the copula function values $C(u, 1-u)$ in Eq. (A.5) and $C(1-u, u)$ in Eq. (A.6) should be the same according to the property of exchangeability. In other words, the traditional symmetric copula models can allow differences between tail coefficients in the *lower-lower* and *upper-upper* domain (as shaded by the yellow color in Fig. A.1), but could not allow any differences between tail coefficients in the *lower-upper* and *upper-lower* domain (as shaded by the red color in Fig. A.1). For instance, if the *lower-upper* and *upper-lower* tail dependence coefficients of a bivariate copula are different (e.g. $\lambda_{1|2}^{u,l} \neq \lambda_{1|2}^{l,u}$), that copula would be considered as an asymmetric one.

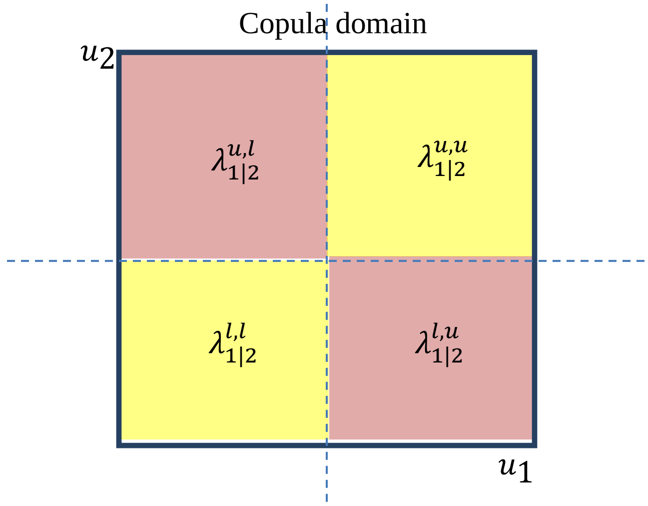


Figure A.1 Tail dependences in the copula domain

References

Beer, M., Zhang, Yi, Tong Quek, S; Phoon, KK (2013). Reliability analysis with scarce information: Comparing alternative approaches in a geotechnical engineering context, Structural Safety, 41, 1-10.

Bond, A. J., Schuppener, B., Nikolova, B., Dimova, S., Pinto, A. V., Orr, T. L., & Scarpelli, G. (2016). Eurocode 7 geotechnical design: Worked examples. Luxembourg: Publications Office.

Charpentier, A., Fougères, A. L., Genest, C., &Nešlehová, J. G. (2014). Multivariate archimax copulas. Journal of Multivariate Analysis, 126, 118-136.

Ching, J. and Phoon, K. K. (2012). Modeling parameters of structured clays as a multivariate normal distribution. Canadian Geotechnical Journal, 49(5), 522-545.

Ching, J., Phoon, K. K., and Chen, C. H. (2014). Modeling CPTU parameters of clays as a multivariate normal distribution. Canadian Geotechnical Journal, 51(1), 77-91.

Ching, Jianye, and Kok-KwangPhoon. (2014). Correlations among some clay parameters — the multivariate distribution. Review of. Canadian Geotechnical Journal 51 (6):686-704. doi: 10.1139/cgj-2013-0353.

D'Ignazio, M., Phoon, K. K., Tan, S. A. and Länsivaara, T. T. (2016). Correlations for undrained shear strength of Finnish soft clays. Canadian Geotechnical Journal, 53(10), 1628-1645.

Fan, Y., & Patton, A. J. (2014). Copulas in econometrics. Annu. Rev. Econ., 6(1), 179-200.

Frank, R. (2004). Designers' guide to EN 1997-1 Eurocode 7: Geotechnical design-General rules. Thomas Telford.

Genest, C., &Nešlehová, J. G. (2013). Assessing and modeling asymmetry in bivariate continuous data. In Copulae in Mathematical and Quantitative Finance (pp. 91-114). Springer, Berlin, Heidelberg.

Genest, C., &Rivest, L. P. (1993). Statistical inference procedures for bivariate Archimedean copulas. Journal of the American statistical Association, 88(423), 1034-1043.

Grimaldi, S., &Serinaldi, F. (2006). Asymmetric copula in multivariate flood frequency analysis. Advances in Water Resources, 29(8), 1155-1167.

He, L., Lu, Z., & Li, X. (2018). Failure-mode importance measures in structural system with multiple failure modes and its estimation using copula. Reliability Engineering & System Safety, 174, 53-59.

Hutchinson, T. P. T. P., & Lai, C. D. (1990). Continuous bivariate distributions emphasising applications (No. 04; QA278, H8.).

Joe, H. (2014). Dependence modeling with copulas. CRC Press.

Khoudraji, A. (1995). Contributions al'etude des copules et a la modelisation de valeurs extremes bivariees.Ph.D. Thesis, University Laval Quebec, Canada.

Klement, E. P., &Mesiar, R. (2006). How non-symmetric can a copula be?.CommentationesMathematicaeUniversitatisCarolinae, 47(1), 141-148.

Li, D. Q., Zhang, L., Tang, X. S., Zhou, W., Li, J. H., Zhou, C. B., &Phoon, K. K. (2015). Bivariate distribution of shear strength parameters using copulas and its impact on geotechnical system reliability. Computers and Geotechnics, 68, 184-195.

Li, Di.Q., Tang, X.S., Zhou, C.B. &Phoon, K.K. (2012). Uncertainty analysis of correlated non-normal geotechnical parameters using Gaussian copula. Review of. Science China Technological Sciences 55 (11):3081-9. doi: 10.1007/s11431-012-4937-z.

Liebscher, E. (2008). Construction of asymmetric multivariate copulas. Journal of Multivariate analysis, 99(10), 2234-2250.

Mazo, G., Girard, S., & Forbes, F. (2015). A class of multivariate copulas based on products of bivariate copulas. Journal of Multivariate Analysis, 140, 363-376.

Mesiar, R., &Najjari, V. (2014). New families of symmetric/asymmetric copulas. Fuzzy Sets and Systems, 252, 99-110.

Nelsen, R. B. (2006). An Introduction to Copulas. Springer New York.

Noh, Y., Choi, K. K., & Du, L. (2009). Reliability-based design optimization of problems with correlated input variables using a Gaussian Copula. Structural and multidisciplinary optimization, 38(1), 1-16.

Phoon, K. K., &Ching, J. (2015). Characterization of geotechnical variability–a multivariate perspective. Analytical Methods in Petroleum Upstream Applications, 61.

Phoon, K. K., &Ching, J. (Eds.). (2014). Risk and reliability in geotechnical engineering. CRC Press.

Phoon, K.K, and Kulhawy, F. (1999). Characterization of geotechnical variability. Review of. Canadian Geotechnical Journal 36 (4):612-24. doi: 10.1139/t99-038.

Pinheiro Branco, L.; Topa Gomes, A: Silva Cardoso, A; and Santos Pereira, C.(2014). "Natural Variability of Shear Strength in a Granite Residual Soil from Porto." Geotechnical and Geological Engineering 32 (4):911-22. doi: 10.1007/s10706-014-9768-1.

Robertson, P. K. (2009). Interpretation of cone penetration tests—a unified approach. Canadian geotechnical journal, 46(11), 1337-1355.

Salvadori, G., & De Michele, C. (2007). On the use of copulas in hydrology: theory and practice. Journal of Hydrologic Engineering, 12(4), 369-380.

Salvadori, G., De Michele, C., Kottegoda, N. T., & Rosso, R. (2007). Extremes in nature: an approach using copulas (Vol. 56). Springer Science & Business Media.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. Publications de l'Institut de Statistique de l'Université de Paris.

Tang, X. S., Li, D. Q., Zhou, C. B., &Phoon, K. K. (2015). Copula-based approaches for evaluating slope reliability under incomplete probability information. Structural Safety, 52, 90-99.

Tang, Xiao-Song, Dian-Qing Li, Guan Rong, Kok-KwangPhoon, and Chuang-Bing Zhou. (2013). Impact of copula selection on geotechnical reliability under incomplete probability information. Review of. Computers and Geotechnics 49 (Supplement C):264-78. doi: https://doi.org/10.1016/j.compgeo.2012.12.002.

Trivedi, P. K., & Zimmer, D. M. (2007). Copula modeling: an introduction for practitioners. Foundations and Trends in Econometrics, 1(1), 1-111.

Wang, C., Zhang, H., & Li, Q. (2017). Reliability assessment of aging structures subjected to gradual and shock deteriorations. Reliability Engineering & System Safety, 161, 78-86.

Wang, F., & Li, H. (2018). System reliability under prescribed marginals and correlations: Are we correct about the effect of correlations?. Reliability Engineering & System Safety, 173: 94-104.

Wu, X. Z. (2013). Trivariate analysis of soil ranking-correlated characteristics and its application to probabilistic stability assessments in geotechnical engineering problems. Soils and Foundations, 53(4), 540-556.

Zhang, Y, Gomes, A.T., Beer, M., Neumann, I., Nackenhorst, U. & Kim, C.W. (2018b) Modeling asymmetric dependences among multivariate soil data for the geoscience analysis - the asymmetric copula approach. CATENA, under review.

Zhang, Y., & Lam, J. S. L. (2016). A copula approach in the point estimate method for reliability engineering. Quality and Reliability Engineering International, 32(4), 1501-1508.

Zhang, Y., Kim, C. W., Beer, M., Dai, H., & Soares, C. G. (2018a). Modeling multivariate ocean data using asymmetric copulas. Coastal Engineering, 135, 91-111.

Zhang, Y. (2018). Investigating dependencies among oil price and tanker market variables by copula-based multivariate models. Energy, 161, 435-446.

Zhang, Y., Beer, M., & Quek, S. T. (2015). Long-term performance assessment and design of offshore structures. Computers & Structures, 154, 101-115.